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Some Majorization Inequalities for Functions of
Exchangeable Random Variables

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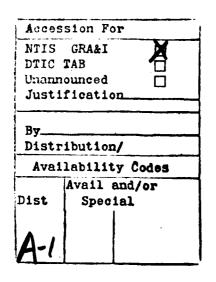Schur-concave functions, moment
inequalities.

Abstract

This paper contains inequalities for the expectations
of permutation-invariant concave functions of the partial
sums of nonnegative exchangeable random variables. Two
majorization inequalities are derived, and an application in
reliability theory is discussed.

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1. Introduction and Summary

For fixed $n > 1$ let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ denote an n-dimensional random vector with density function $f(\underset{\sim}{x})$ that is absolutely continuous w.r.t. the Lebesgue measure or the product measure of counting measures. $X_1, \ldots, X_n$ are said to be exchangeable[+] if f is invariant under permutations of its arguments. This paper concerns majorization inequalities for the expectations of certain functions of $X_1, \ldots, X_n$.

The notion of majorization defines a partial ordering of the diversity of the components of vectors. Let $\underset{\sim}{a} = (a_1, \ldots, a_n)$, $\underset{\sim}{b} = (b_1, \ldots, b_n)$ be two n-dimensional vectors and let $a_{(1)} \geq \ldots \geq a_{(n)}$, $b_{(1)} \geq \ldots \geq b_{(n)}$ denote their ordered components. $\underset{\sim}{a}$ is said to majorize $\underset{\sim}{b}$, (in symbols $\underset{\sim}{a} \succ \underset{\sim}{b}$), if

$$\sum_1^m a_{(i)} \geq \sum_1^m b_{(i)} \qquad \text{for } m = 1, \ldots, n-1$$

and $\sum_1^n a_i = \sum_1^n b_i$. It is known that $\underset{\sim}{a} \succ \underset{\sim}{b}$ iff there exists a doubly stochastic matrix Q such that $\underset{\sim}{b} = \underset{\sim}{a}Q$, i.e., $\underset{\sim}{b}$ is an "average" of $\underset{\sim}{a}$. A function $\psi: R^n \to R$ is said to be a Schur-concave function if $\underset{\sim}{a} \succ \underset{\sim}{b}$ implies $\psi(\underset{\sim}{a}) \leq \psi(\underset{\sim}{b})$. For a full treatment of majorization and Schur functions, see Marshall and Olkin (1979).

In an earlier paper Marshall and Proschan (1965) proved

[+]More precisely, $X_1, \ldots, X_n$ are <u>finitely</u> exchangeable instead of exchangeable. For the minor distinction between finite exchangeability and exchangeability see e.g., Tong ((1980), p. 96.

the following inequality: Let $X_1,\ldots,X_n$ be exchangeable and let $\phi: R^n \to R$ be Borel measurable, permutation invariant and concave. If $a_1,\ldots,a_n$ are more diverse than $b_1,\ldots,b_n$ in the sense of majorization, i.e., if $\underset{\sim}{a} \succ \underset{\sim}{b}$, then $E\phi(b_1 X_1,\ldots,b_n X_n) \geq E\phi(a_1 X_1,\ldots,a_n X_n)$ provided the expectations exist. This inequality yields a number of useful results and implies many previously-known results as special cases (see, e.g., Corollaries 1-3 in their paper). In this paper, we prove some related results and discuss their applications. The results (Theorems 1 and 2) involve the expectations of functions of partial sums of exchangeable random variables, and depend on the notion of majorization in a different manner. For fixed $k < n$, let $\underset{\sim}{r} = (r_1,\ldots,r_k)$ be a vector of positive integers such that $\overset{k}{\underset{1}{\Sigma}} r_i = n$. Let $X_1,\ldots,X_n$ be exchangeable random variables and let $\underset{\sim}{Y}_{\underset{\sim}{r}} = (Y_1^{(\underset{\sim}{r})},\ldots,Y_k^{(\underset{\sim}{r})})$ denote a k-dimensional random vector such that

$$Y_1^{(\underset{\sim}{r})} = \overset{r_1}{\underset{1}{\Sigma}} X_i, \quad Y_2^{(\underset{\sim}{r})} = \overset{r_1+r_2}{\underset{r_1+1}{\Sigma}} X_i,\ldots,Y_k^{(\underset{\sim}{r})} = \overset{n}{\underset{r_1+\ldots+r_{k-1}+1}{\Sigma}} X_i;$$

that is, $Y_j^{(\underset{\sim}{r})}$ is the sum of r_j of such X_i's and $Y_1^{(\underset{\sim}{r})},\ldots,Y_k^{(\underset{\sim}{r})}$ do not contain any common elements. Let $\underset{\sim}{s} = (s_1,\ldots,s_k)$ denote another such vector and $\underset{\sim}{Y}^{(s)}$ be defined similarly. Let $\phi(\underset{\sim}{y}) = \phi(y_1,\ldots,y_k)$ denote any real-valued function that is permutation invariant and concave for every fixed $\overset{k}{\underset{1}{\Sigma}} y_i$. We show that (Theorem 1) if $\underset{\sim}{s} \succ \underset{\sim}{r}$ and if the X_i's are non-negative exchangeable random variables, then $E\phi(\underset{\sim}{Y}^{(\underset{\sim}{r})}) \geq E\phi(\underset{\sim}{Y}^{(\underset{\sim}{s})})$. The needs for considering such a random vector $\underset{\sim}{Y}$ and for

studying inequalities of this type arise from certain applications. One such application concerns the optimal arrangement policy for parallel and series systems in reliability theory, and is given in Section 4. In Theorem 2 we show that, by imposing an additional condition on the joint density f, the same inequality holds for all Schur-concave functions ϕ.

Since the theorems apply to nonnegative random variables only, a natural question is whether or not the same statements hold for random varaibles which may take negative values. We show in Section 3 that the answer is negative even for i.i.d. normal variables.

2. The Main Results

For the theorems stated in this section, the density function f of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is assumed to be absolutely continuous w.r.t. the Lebesgue measure or the product measure of the counting measures. The proofs will be given for the former. For the product of counting measures, simply change the integral signs to summation signs.

Theorem 1. If (i) f is permutation invariant and f = 0 if any $x_i < 0$ (i = 1,...,n), (ii) $\phi(y_1, \ldots, y_k)$ is a concave function for every fixed $\sum_1^k y_j$, and (iii) $\underset{\sim}{s} \succ \underset{\sim}{r}$, then

$$E\phi(Y_1^{(\underset{\sim}{r})}, \ldots, Y_k^{(\underset{\sim}{r})}) \geq E\phi(Y_1^{(\underset{\sim}{s})}, \ldots, Y_k^{(\underset{\sim}{s})}) \qquad (2.1)$$

holds provided that the expectations exist.

<u>Proof</u>. It is well-known (Marshall and Olkin (1979), Chapter 2) that for simplicity one may assume that

$$s_1 > r_1 \geq r_2 > s_2 \equiv t, \qquad r_1 + r_2 = s_1 + s_2 \equiv d$$

and $r_j = s_j$ for $j = 3,\ldots,k$. Let us define

$$z_1 = \sum_1^t X_i, \qquad z_2 = \sum_{s_1+1}^d X_i$$

and $Y_j = Y_j^{(\underset{\sim}{r})} = Y_j^{(\underset{\sim}{s})}$ for $j = 3,\ldots,k$. Let

$$g(z_1,z_2) = g(z_1,z_2 | \underset{\sim}{x}_0, y_3,\ldots,y_k) \qquad (2.2)$$

denote the conditional density of (z_1,z_2) given $\underset{\sim}{X}_0 \equiv (X_{t+1},\ldots,X_{s_1}) = \underset{\sim}{x}_0$ and $Y_j = y_j$ ($j = 3,\ldots,k$). Then it is easy to check that $g(z_1,z_2) = g(z_2,z_1)$ and

$$E\phi(Y_1^{(\underset{\sim}{r})},\ldots,Y_k^{(\underset{\sim}{r})}) = E[\iint \phi(z_1+u_1,z_2+u_2,y_3,\ldots,y_k)g(z_1,z_2)dz_1dz_2]$$

$$= E[\iint_{z_1 \geq z_2} \phi(z_1+u_1,z_2+u_2,y_3,\ldots,y_k)g(z_1,z_2)dz_1dz_2$$

$$+ \iint_{z_1 < z_2} \phi(z_1+u_1,z_2+u_2,y_3,\ldots,y_k) g(z_1,z_2)dz_1dz_2]$$

$$= E[\iint_{z_1 \geq z_2} \{\phi(z_1+u_1,z_2+u_2,y_3,\ldots,y_k)$$

$$+ \phi(z_1+u_2,z_2+u_1,y_3,\ldots,y_k)\}\cdot g(z_1,z_2)dz_1dz_2]$$

where $(u_1,u_2) = (\sum_{i=t+1}^{r_1} x_i, \sum_{i=r_1+1}^{s_1} x_i)$. Now let $(v_1,v_2) = (\sum_{i=t+1}^{s_1} x_i, 0)$. Since $x_i \geq 0$, there exists an $\alpha = \dfrac{u_1}{v_1} \in [0,1]$ which satisfies

$$(z_1+u_1, z_2+u_2) = \alpha(z_1+v_1, z_2+v_2) + (1-\alpha)(z_1+v_2, z_2+v_1),$$

$$(z_1+u_2, z_2+u_1) = (1-\alpha)(z_1+v_1, z_2+v_2) + \alpha(z_1+v_2, z_2+v_1)$$

for every point in $\{(z_1, z_2) : z_1 \geq z_2\}$. Thus, for every fixed $(\underset{\sim}{x}_0, y_3, \ldots, y_k)$ and every such (z_1, z_2),

$$\phi(z_1+u_1, z_2+u_2, y_3, \ldots, y_k) + \phi(z_1+u_2, z_2+u_1, y_3, \ldots, y_k)$$

$$\geq \alpha\phi(z_1+v_1, z_2+v_2, y_3, \ldots, y_k) + (1-\alpha)\phi(z_1+v_2, z_2+v_1, y_3, \ldots, y_k)$$

$$+ (1-\alpha)\phi(z_1+v_1, z_2+v_2, y_3, \ldots, y_k) + \alpha\phi(z_1+v_2, z_2+v_1, y_3, \ldots, y_k).$$

Consequently one has

$$E\phi(Y_1^{(\underset{\sim}{r})}, \ldots, Y_k^{(\underset{\sim}{r})})$$

$$\geq E[\iint_{z_1 \geq z_2} \{\phi(z_1+v_1, z_2+v_2, y_3, \ldots, y_k) + \phi(z_1+v_2, z_2+v_1, y_3, \ldots, y_k)\}$$

$$\times g(z_1, z_2) dz_1 dz_2]$$

$$= E\phi(Y_1^{(\underset{\sim}{s})}, \ldots, Y_k^{(\underset{\sim}{s})})$$

as to be shown. □

In the next theorem we weaken the condition on ϕ to be any measurable Schur-concave function, and impose a stronger condition on the conditional density g.

Theorem 2. If (i) f is permutation invariant, f = 0 for any $x_i < 0$ (i = 1,...,n), and such that the conditional density $g(z_1, z_2)$ defined in (2.2) is a Schur-concave function of (z_1, z_2) for every fixed $(\underset{\sim}{x}_0, y_3, \ldots, y_k)$ and every t > 0,

(ii) $\phi(y_1,\ldots,y_k)$ is a Borel-measurable Schur-concave function, and (iii) $\underset{\sim}{s} \succ \underset{\sim}{r}$, then (2.1) holds provided the expectations exist.

<u>Proof</u>. We shall follow the notation developed in the proof of Theorem 1 and compare $E\phi(\underset{\sim}{Y}^{(\underset{\sim}{r})})$ with $E\phi(\underset{\sim}{Y}^{(\underset{\sim}{s})})$ for $\underset{\sim}{s} \succ \underset{\sim}{r}$. Again for simplicity assume that $s_1 > r_1 \geq r_2 > s_2$ and $r_j = s_j$ for $j > 2$. Then one can write

$$\Delta = E[\phi(Y_1^{(\underset{\sim}{r})},Y_2^{(\underset{\sim}{r})},Y_3,\ldots,Y_k) - \phi(Y_1^{(\underset{\sim}{s})},Y_2^{(\underset{\sim}{s})},Y_3,\ldots,Y_k)]$$

$$= E[\iint \{\phi'(z_1+u_1,z_2+u_2)-\phi'(z_1+u_1+u_2,z_2)\}g(z_1,z_2)dz_1dz_2]$$

where for notational convenience $\phi'(y_1,y_2)$ stands for $\phi(y_1,y_2,y_3,\ldots,y_k)$ and g is the conditional density of (z_1,z_2). It is straightforward to verify that, after following the same steps as in the proof of Theorem J.1 in Marshall and Olkin (1979, p. 100), one has

$$\Delta = E[\iint_{z_1 \geq z_2} \{\phi'(z_1,z_2+u_2) - \phi'(z_1+u_2,z_2)\}$$
$$\times \{g(z_1-u_1,z_2) - g(z_1,z_2-u_1)\}dz_1dz_2].$$

Since ϕ' and g are Schur-concave functions and $u_i > 0$ $(i = 1,2)$, one has

$$(z_1+u_2,z_2) \succ (z_1,z_2+u_2),$$

$$(z_1,z_2-u_1) \succ (z_1-u_1,z_2).$$

Thus $\Delta \geq 0$. □

Remark. Proschan and Sethuraman (1977) previously proved that if $X_1, \ldots, X_n$ are i.i.d. nonnegative random variables with a common density $h(x)$ that is log-concave, then the conclusion in Theorem 2 holds. Their proof depends on an application of the main theorem in their paper and on a TP_2 property of the convolution of log-concave densities given in Karlin and Proschan (1960). It is noted here that their result now follows immediately from Theorem 2. This is so because if $X_1, \ldots, X_n$ are i.i.d. random variables with a common density that is log-concave, then $\sum_{1}^{s_2} X_i$ and $\sum_{s_1+1}^{s_1+s_2} X_i$ are independent random variables with a common density that is also log-concave (see e.g., Das Gupta (1973, Theorem 4.2)). Consequently, the joint density of $(\sum_{1}^{s_1} X_i, \sum_{s_1+1}^{s_1+s_2} X_i)$ is a Schur-concave function and Theorem 2 applies.

In most applications, the assumption on the Schur-concavity of the conditional density $g(z_1, z_2)$ of (Z_1, Z_2) is not easy to verify. It is clear that the assumption holds if the following conjecture concerning the convolution of Schur-concave random variables is true:

Conjecture 1. For $n = mk$ and $\underset{\sim}{X} = (X_1, \ldots, X_n)$ let

$$Z_j = \sum_{(j-1)m+1}^{jm} X_i, \quad j = 1, 2, \ldots, k.$$

If the joint density of $\underset{\sim}{X}$ is a Schur-concave function of $\underset{\sim}{x}$ for $\underset{\sim}{x} \in R^n$, then the joint density of $\underset{\sim}{Z} = (Z_1, \ldots, Z_k)$ is a Schur-concave function of $\underset{\sim}{z}$ for $\underset{\sim}{z} \in R^k$ for all positive integers k and m.

It is not yet known to us whether this conjecture is true or not for the continuous case. However, the following counterexample shows that at least it is not true for the discrete case.

Example 1. Consider $k = m = 2$, and assume that (X_1, X_2, X_3, X_4) takes only integer values 0,1,2,3. Let $Z_1 = X_1 + X_2$, $Z_2 = X_3 + X_4$. Then $P[Z_1 = 4, Z_2 = 2]$ is the probability of the set of the following points:

$$(3,1,1,1), \quad (1,3,1,1), \quad (2,2,1,1), \quad (2,2,2,0) \quad (2,2,0,2)$$
$$(3,1,2,0), \quad (3,1,0,2), \quad (1,3,2,0), \quad (1,3,0,2)$$

Similarly $P[Z_1 = Z_2 = 3]$ is the probability of the set consisting of

$$(2,1,2,1), \quad (2,1,1,2), \quad (1,2,2,1), \quad (1,2,1,2),$$
$$(2,1,3,0), \quad (2,1,0,3), \quad (1,2,3,0), \quad (1,2,0,3),$$
$$(3,0,2,1), \quad (3,0,1,2), \quad (0,3,2,1), \quad (0,3,1,2),$$
$$(3,0,3,0), \quad (3,0,0,3), \quad (0,3,3,0), \quad (0,3,0,3).$$

If the joint density of (X_1, X_2, X_3, X_4) takes values 1/14 for each of the points $(3,1,1,1)$, $(2,2,2,0)$, $(2,2,1,1)$ and all of their permutations, and zero otherwise, then it is a Schur-concave function on the product of integer spaces, and one has

$$P[Z_1 = 4, Z_2 = 2] = \frac{5}{14} > \frac{4}{14} = P[Z_1 = 3, Z_2 = 3].$$

A related problem to Conjecture 1 is the study of a subclass of Schur-concave densities. One such subclass is the

class of all log-concave densities which are permutation invariant. This consideration leads us to the next conjecture.

Conjecture 2. In Conjecture 1 if the joint density of X is log-concave and permutation invariant, then the joint density of Z is Schur-concave (or even log-concave) and permutation invariant.

3. An Example for Random Variables which are Not Nonnegative

In view of the fact that in Marshall and Proschan (1965) the random variables are not necessarily nonnegative, it might be tempting to think that results similar to our Theorems 1 and 2 might also hold when the condition that $X_i \geq 0$ a.s. is removed. In the following we show that this is not true even for i.i.d. normal variables.

Consider, for $n = 2m$, independent normal variables $X_1, \ldots, X_n$ with means zero and variances one. For $t \leq m$ consider

$$Y_1 = \sum_{i=1}^{t} X_i, \qquad Y_2 = \sum_{t+1}^{n} X_i,$$

and denote $U = Y_1 - Y_2$, $V = Y_1 + Y_2 = \sum_1^n X_i$. Then (U,V) has a bivariate normal distribution with means zero, variances n, and correlation $2t/n - 1$. Thus the conditional distribution of U given $V = v$ is normal with mean $\frac{1}{n}(2t-n)v$ and variance $\sigma^2_{U|V=v} = 4t(n-t)/n$. Now choose $n = 4$ and for $\varepsilon > 0$ define

$$\phi(y_1, y_2) = \begin{cases} -(y_1 - y_2)^2 & \text{for } 0 \le |y_1 + y_2| \le \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ is permutation invariant and concave in (y_1, y_2) for fixed $y_1 + y_2$. It follows that

$$E[\{\phi(X_1, \sum_2^4 X_i) - \phi(\sum_1^2 X_i, \sum_3^4 X_i)\} | \sum_1^4 X_i = 0]$$

$$= -\text{Var}((X_1 - \sum_2^4 X_i) | V = \sum_1^4 X_i = 0) + \text{Var}((\sum_1^2 X_i - \sum_3^4 X_i) | V = 0)$$

$$= -4(4-1)/4 + 4 \times 2(4-2)/4 = 1 > 0.$$

By continuity, there exists an $\epsilon > 0$ small enough such that

$$E[\phi(X_1, \sum_2^4 X_i) - \phi(\sum_1^2 X_i, \sum_3^4 X_i)]$$

$$= \int_{-\epsilon}^{\epsilon} E[\{\phi(X_1, \sum_2^4 X_i) - \phi(\sum_1^2 X_i, \sum_3^4 X_i) | \sum_1^4 X_i = v]dP[\sum_1^4 X_i \le v] > 0.$$

4. An Application in Reliability Theory

In this section we state an application of Theorem 1 in reliability theory. Consider n exchangeable components with life lengths $X_1, \ldots, X_n$ which are obviously nonnegative. If the components are manufactured independently, then the joint density f of the X_i's is the product of the common marginal density; otherwise, if they are manufactured under the influence of some common factors or a common environment, then it is well-known that f is a mixture and the random variables are conditionally i.i.d. In either case f is permutation invariant.

Now suppose that a system consists of k subsystems, and

that the j-th subsystem, consisting of $r_j \geq 1$ such components, is required to operate properly with one component in operation and the others in a standby capacity $(j = 1,\ldots,k)$. Then the life length Y_j of the j-th sybsystem is $\sum_{r_1+\ldots+r_{j-1}+1}^{r_1+\ldots+r_j} X_i$. Now let $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(k)}$ denote the order statistics of $Y_1,\ldots,Y_k$ and $\underline{r} = (r_1,\ldots,r_k)$ be an allocation vector such that $r_j \geq 1$ and $\sum_1^k r_j = n$. When the subsystems are connected in series, then the life length of the system is $Y_{(1)}$. On the other hand if they are connected in parallel, then it is $Y_{(k)}$. Now for fixed $c_i \geq 0$, $\sum_1^k c_j Y_{(j)}$ is permutation invariant and is a concave function (a convex) function of $(y_1,\ldots,y_k)$ given $\sum_1^k y_j$ if $c_1 \geq \ldots \geq c_k$ (if $c_1 \leq \ldots \leq c_k$). Consequently, Theorem 1 provides a partial ordering for the expected life length of the system for series and parallel systems. In particular, for series systems the optimal allocation policy is such that $|r_j - r_{j'}| \leq 1$ for all $j \neq j'$, and for parallel systems an optimal policy is that $r_1 = n-k+1$ and $r_2 = \ldots = r_k = 1$.

References

Das Gupta, S. (1973). S-unimodal functions: Related inequa-
lities and statistical applications. Sankhyā, Ser. B, 38,
301-314.

Karlin, S. and Proschan, F. (1960). Pólya type distributions
of convolutions. Ann. Math. Statist., 31, 721-736.

Marshall, A. W. and Olkin, I. (1979). Inequalities: Theory
of Majorization and Its Applications. New York: Academic
Press.

Marshall, A. W. and Proschan, F. (1965). An inequality for
convex functions involving majorization. J. Math. Anal.
Appl., 12, 87-90.

Proschan, F. and Sethuraman, J. (1977). Schur functions in
statistics. I. The preservation theorem. Ann. Math.
Statist., 5, 256-262.

Tong, Y. L. (1980). Probability Inequalities in Multivariate
Distributions. New York: Academic Press.

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